Linear Algebra

Unit III

Prove that $V_n(R)$ is a real inner product space with inner product defined by

$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

Solution: Let $x, y, z \in V_n(R)$, and $\alpha \in R$

$x = \cancel{(x_1, y_1} (x_1, x_2, \dots x_n)$

$y = (y_1, y_2, \dots y_n)$

$z = (z_1, z_2, \dots z_n)$

(i)

$\langle x+y, z\rangle = (x_1+y_1) z_1 + (x_2+y_2) z_2 \dots + (x_n+y_n) z_n$

$= (x_1 z_1 + x_2 z_2 + \dots + x_n z_n) + (y_1 z_1 + y_2 z_2 + \dots + y_n z_n)$

$= \langle x, z\rangle + \langle y, z\rangle$

(ii) $\langle \alpha x, y\rangle = (\alpha x_1) y_1 + (\alpha x_2) y_2 + \dots + (\alpha x_n) y_n$

$= \alpha (x_1 y_1) + \alpha (x_2 y_2) + \dots + \alpha (x_n y_n)$

$= \alpha (x_1 y_1 + x_2 y_2 + \dots + x_n y_n)$

$= \alpha \langle x, y\rangle$

(iii) $\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$= y_1 x_1 + y_2 x_2 + \dots + y_n x_n$

$\langle x, y\rangle = \langle y, x\rangle$

(iv) $\langle x, x \rangle = x_1 x_1 + x_2 x_2 + \cdots + x_n x_n$

$= x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0.$

$\therefore \langle x, x \rangle = 0$ iff $x = 0$

$\therefore V_n(R)$ is a Real inner Product space.

——— xxx

2. $V_n(C)$ is a Complex inner product space with inner product defined by

$\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}$

Sol.: Let $x = (x_1, x_2, \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$, $z = (z_1, z_2, \ldots z_n)$ $\in V_n(C)$

(i) $\langle x+y, z \rangle = (x_1+y_1)\overline{z_1} + (x_2+y_2)\overline{z_2} + \cdots + (x_n+y_n)\overline{z_n}$

$= (x_1\overline{z_1} + x_2\overline{z_2} + x_3\overline{z_3} + \cdots + x_n\overline{z_n}) + (y_1\overline{z_1} + y_2\overline{z_2} + \cdots + y_n\overline{z_n})$

$= \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, y \rangle = (\alpha x_1)\overline{y_1} + (\alpha x_2)\overline{y_2} + \cdots + (\alpha x_n)\overline{y_n}$

$= \alpha x_1 \overline{y_1} + \alpha x_2 \overline{y_2} + \cdots + \alpha x_n \overline{y_n}$

$= \alpha (x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n})$

$= \alpha \langle x, y \rangle$

(iii) $\overline{\langle y, x\rangle} = \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n}$

$= \bar{y}_1\bar{\bar{x}}_1 + \bar{y}_2\bar{\bar{x}}_2 + \cdots + \bar{y}_n\bar{\bar{x}}_n$

$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$

$= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$

$\overline{\langle y, x\rangle} = \langle x, y\rangle$

(iv) $\langle x, x\rangle = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n$

$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$

$\Rightarrow \langle x, x\rangle \geq 0$. and $\langle x, x\rangle = 0$ iff $x = 0$.

$\Rightarrow$ Hence $V_n(C)$ is a Complex inner product space.